MATHEMATICS
HIGHER LEVEL
PAPER 2

Tuesday 8 May 2007 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 23]*

Two planes π_1 and π_2 are represented by the equations

$$\pi_1 : \boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\pi_2 : 2x - y - 2z = 4 .$$

(a) (i) Find $\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

(ii) Show that the equation of π_1 can be written as $x - 2y + 2z = 11$. *[4 marks]*

(b) Show that π_1 is perpendicular to π_2. *[4 marks]*

(c) The line l_1 is the line of intersection of π_1 and π_2.
Find the vector equation of l_1, giving the answer in parametric form. *[5 marks]*

(d) The line l_2 is parallel to **both** π_1 and π_2, and passes through $\mathrm{P}(3, -5, -1)$.
Find an equation for l_2 in Cartesian form. *[3 marks]*

(e) Let Q be the foot of the perpendicular from P to the plane π_2.

(i) Find the coordinates of Q.

(ii) Find PQ. *[7 marks]*

2. *[Maximum mark: 24]*

(a) Using the formula for $\cos(A+B)$ prove that $\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$. *[3 marks]*

(b) Hence, find $\int \cos^2 x \, dx$. *[4 marks]*

Let $f(x) = 4\cos x$ and $g(x) = \sec x$ for $x \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$.
Let R be the region enclosed by the two functions.

(c) Find the **exact** values of the x-coordinates of the points of intersection. *[4 marks]*

(d) Sketch the functions f and g and clearly shade the region R. *[3 marks]*

The region R is rotated through 2π about the x-axis to generate a solid.

(e) (i) Write down an integral which represents the volume of this solid.

(ii) Hence find the **exact** value of the volume. *[10 marks]*

3. *[Total Mark: 26]*

Part A *[Maximum mark: 18]*

The time, T minutes, spent each day by students in Amy's school sending text messages may be modelled by a normal distribution.

30 % of the students spend less than 10 minutes per day.
35 % spend more than 15 minutes per day.

(a) Find the mean and standard deviation of T. *[6 marks]*

The number of text messages received by Amy during a fixed time interval may be modelled by a Poisson distribution with a mean of 6 messages per hour.

(b) Find the probability that Amy will receive exactly 8 messages between 16:00 and 18:00 on a random day. *[3 marks]*

(c) Given that Amy has received at least 10 messages between 16:00 and 18:00 on a random day, find the probability that she received 13 messages during that time. *[5 marks]*

(d) During a 5-day week, find the probability that there are exactly 3 days when Amy receives no messages between 17:45 and 18:00. *[4 marks]*

Part B *[Maximum mark: 8]*

Twenty candidates sat an examination in French. The sum of their marks was 826 and the sum of the squares of their marks was $34\,132$. Two candidates sat the examination late and their marks were a and b. The new mean and variance were calculated, giving the following results:

$$\text{mean} = 42 \text{ and variance} = 32 .$$

Find a set of possible values of a and b. *[8 marks]*

4. *[Total Mark: 21]*

Part A *[Maximum mark: 11]*

(a) Find the probability that a number, chosen at random between 200 and 800 inclusive, will be a multiple of 9. *[3 marks]*

(b) Find the sum of the numbers between 200 and 800 inclusive, which are multiples of 6, but **not** multiples of 9. *[8 marks]*

Part B *[Maximum mark: 10]*

Prove **by induction** that $12^n + 2(5^{n-1})$ is a multiple of 7 for $n \in \mathbb{Z}^+$. *[10 marks]*

5. *[Maximum mark: 26]*

(a) (i) Factorize $t^3 - 3t^2 - 3t + 1$, giving your answer as a product of a linear factor and a quadratic factor.

(ii) Hence find all the **exact** solutions to the equation $t^3 - 3t^2 - 3t + 1 = 0$. *[5 marks]*

(b) Using de Moivre's theorem and the binomial expansion

(i) show that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$;

(ii) write down a similar expression for $\sin 3\theta$. *[7 marks]*

(c) (i) Hence show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

(ii) Find the values of θ, $0^\circ \le \theta \le 180^\circ$, for which this identity is not valid. *[7 marks]*

(d) Using the results from parts (a) and (c), find the **exact** values of $\tan 15^\circ$ and $\tan 75^\circ$. *[7 marks]*